ANALYSIS 2016.

1. Individual

Problem 1.1. Suppose that f is analytic on $\Delta = \{|z| < 1\}$ and $|f(z)| \le 1$ for $z \in \Delta$. If $f(\frac{1}{2}) = f(-\frac{1}{2}) = 0$, then

$$|f(0)| \le \frac{1}{4}.$$

Problem 1.2. Let $a_1 = \sin(x)$ where $x \in (0, \frac{\pi}{2})$. For $n \ge 2$, $a_n = \sin(a_{n-1})$. Show that the series

$$\sum_{n=1}^{\infty} a_n^2$$

is divergent.

Problem 1.3. Let $L^p[0,1]$ be the space of L^p-integrable functions on $[0,1]$ where

$$\|f\|_p = \left(\int_0^1 |f|^p dx \right)^{1/p}, \quad p > 0.$$

Show the $\| \bullet \|_p$ satisfies the parallelogram law if and only if $p = 2$.

Problem 1.4. Find all the bounded solutions of

$$\begin{cases} \Delta u(x,y) = 0, \quad \text{for} \quad (x,y) \in \mathbb{R}^2_+, \\ u(x,0) = \begin{cases} 1, x > 0, \\ 0, x < 0 \end{cases} \end{cases}$$

Problem 1.5. Suppose $f(x) \in C[0,+\infty)$, and for any non-negative real number a, we have

$$\lim_{x \to +\infty} (f(x+a) - f(x)) = 0. \tag{1}$$

Show that there exist $g(x) \in C[0,+\infty)$ and $h(x) \in C^1[0,+\infty)$ such that $f(x) = g(x) + h(x)$ and

$$\lim_{x \to +\infty} g(x) = 0, \quad \lim_{x \to +\infty} h'(x) = 0. \tag{2}$$

Problem 1.6. Let f be a Riemann integrable function on $[-\pi, \pi]$ with Fourier series

$$f(x) \sim \sum_{n=-\infty}^{\infty} a_n e^{inx}. \tag{3}$$

Suppose

$$|a_n| \le \frac{K}{|n|}$$

for some positive constant K and all $n \ne 0$. Show that

$$\left| \sum_{n=-N}^{N} a_n e^{inx} \right| \le \sup_{y \in [-\pi,\pi]} |f(y)| + 2K \tag{4}$$

for all $x \in [-\pi, \pi]$ and all $N \in \mathbb{N}$.